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ZERO-DIVISORS IN COMPLETIONS OF NON-COMMUTATIVE RINGS

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ABSTRACT

We show that it is possible for a regular element of a noncommutative Noetherian ring R to become a zero-divisor in the M-adic completion of R for a maximal ideal M of R.

1. Introduction

Let R be a Noetherian ring with a prime ideal M such that $\bigcap_{n\geq 1} M^n = 0$ and let $\hat{R}$ be the M-adic completion of R. If R is commutative and a is a regular element of R then it is known that a must remain regular in $\hat{R}$, see [10]. This is a consequence of the flatness of $\hat{R}$ as an R-module which in turn is a consequence of the Artin–Rees property. In the non-commutative case it is well known that the Artin–Rees property need not hold and it is also known, see [2], that $\hat{R}$ need not be flat as an R-module. The main purpose of this note is to present, in Section 2, an example in which M is generated by a normalizing sequence of elements, the first of which becomes a zero-divisor in $\hat{R}$ despite being regular in R. In this example R/M is Artinian and $\hat{R}$ is Noetherian by [8, Theorem 4.2]. The example is related to the one we constructed in [5, Section 3], where $\hat{R}$ is not Noetherian although the intersection of the symbolic powers of M is zero, and, as in [5], the main difficulty in checking the details lies in the verification that $\bigcap_{n\geq 1} M^n = 0$. Although the approach used for this in [5] can be taken here,

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with greater technical difficulty, we offer an alternative approach, which may be of independent interest, based on embedding a certain commutative ring in a ring of formal power series.

Our notation for completions will be standard, as used, for example, in [2, 4, 5, 6, 8]. When dealing with an Ore extension or ring of formal differential operators $A[\theta;\delta]$ over a ring with derivation δ, we shall write coefficients on the left; thus $\theta a = a\theta + \delta(a)$ for all $a \in A$. Any unexplained terminology will be as in [9].

1.1. The result below, in the spirit of [4, Proposition 2] and [5, Proposition 2], suggests a strategy for constructing an example with the properties described in 1.0 and offers a different perspective on the failure of flatness for noncommutative completions to that in [2].

PROPOSITION: *Let R be a ring with a maximal ideal M such that $\bigcap_{n\geq 1} M^n = 0$. Let a_1, a_2 be a normalizing sequence of elements of R contained in M such that $a_1M = Ma_1$ but $a_2M \not\subseteq Ma_2 + a_1R$. Let $\hat{R}$ be the M-adic completion of R. Then $a_2 \in a_1\hat{R}$. Consequently, if $a_2 \notin a_1R$ then $\hat{R}$ is not flat as a right R-module.*

Proof: Let $N = \{r \in R : ra_2 \in a_2M + a_1R\}$. Then N is an ideal of R and is not contained in M. Thus there exist $m \in M$ and $n \in N$ such that $m + n = 1$. Write $na_2 = a_2m' + a_1r$ and $ma_1 = a_1m''$ where $m', m'' \in M$ and $r \in R$. We recursively construct two sequences $\{r_i\}_{i\geq 1}$ and $\{m_i\}_{i\geq 1}$ as follows. Set $m_1 = a_2 - a_1r, r_1 = r$; thus $a_2 = m_1 + a_1r_1$. Let $i > 1$ and suppose that r_k and m_k have been chosen for $1 \leq k < i$ and that $m_k \in M^k$ and $a_2 = m_k + a_1r_k$ for $1 \leq k < i$. Then

$$\begin{aligned} a_2 &= ma_2 + na_2 \\ &= m(m_{i-1} + a_1r_{i-1}) + a_2m' + a_1r \\ &= m(m_{i-1} + a_1r_{i-1}) + (m_{i-1} + a_1r_{i-1})m' + a_1r \\ &= (mm_{i-1} + m_{i-1}m') + a_1(m''r_{i-1} + r_im' + r). \end{aligned}$$

Set $m_i = mm_{i-1} + m_{i-1}m' \in M^i$ and set $r_i = m''r_{i-1} + r_im' + r$. Thus $a_2 = m_i + a_1r_i$. Note that $r_2 - r_1 \in M$ and that, if $i > 2$, then $r_i - r_{i-1} = m''(r_{i-1} - r_{i-2}) + (r_{i-1} - r_{i-2})m'$. It follows that $r_{i+1} - r_i \in M^i$ for all $i \geq 1$ and hence that $\{r_i\}_{i\geq 1}$ is a Cauchy sequence in the M-adic topology. Let $\hat{r}$ be the limit of $\{r_i\}_{i\geq 1}$ in $\hat{R}$. Then $a_2 - a_1\hat{r}$ is the M-adic limit of $\{a_2 - a_1r_i\}_{i\geq 1}$ which is zero as $a_2 - a_1r_i = m_i \in M^i$. Thus $a_2 = a_1\hat{r} \in a_r\hat{R}$. If $a_2 \notin a_1R$ then $\hat{R}$ cannot be flat by [12, I.10.7]. ∎

1.2 Remark: If, in the situation of 1.1, $s, t \in R$ are such that $a_1 s = a_2 t$ then, with $a_2 = a_1 \hat{r}$ in $\hat{R}, a_1(s - \hat{r}t) = 0$. This is the basis of our construction, in Section 2, where completion does not preserve regularity.

1.3 QUESTION: *In the situation of 1.1, it is not hard to check that $a_1\hat{R} = \hat{R}a_1$. It would be interesting to know whether this ideal must be closed in $\hat{R}$. If R is Noetherian and R/M is Artinian then this is true by [3, Lemma 3] or [2, Corollary 5].*

1.4 Example: We present an example where the conditions of 1.1 are satisfied and it is easy to explicitly express a_2 as a multiple of a_1 in $\hat{R}$.

Let k be a field of characteristic zero, let $A = k[x, z]$ be the commutative polynomial ring in two indeterminates and let δ be the k-derivation $z^2\partial/\partial z + (x + z^2)\partial/\partial x$. Let R be the Ore extension $A[\theta; \delta]$ and observe that z, x, θ is a normalizing sequence generating a maximal ideal M of R with $R/M \simeq k$. Observe that $zR = Rz$ but that $\theta x = x(\theta + 1) + z^2$ so that, since $\theta + 1 \notin M$ and x is regular modulo zR, the conditions of 1.1 are satisfied with $a_1 = z$ and $a_2 = x$. It is not obvious that $\bigcap_{n\geq 1} M^n = 0$. One approach is to adapt the calculations given in [5, Section 3] by substituting 1 for y where appropriate. However the approach used in Section 2 will also be applicable and gives a shorter proof, see 2.11.

Since $\theta \in M, \delta(M^n) \subseteq M^{n+1}$ for all n. Hence $\delta^n(x) \in M^{n+1}$ for all n. But $\delta^n(x) = x + z^2 + 2z^3 + \cdots + n!z^{n+1}$. It follows that if $\ell = -(z + 2z^2 + \cdots + n!z^n + \cdots)$ is the M-adic limit of the Cauchy sequence $\{-(z + 2z^2 + \cdots + n!z^n)\}_{n\geq 1}$ then in $\hat{R}, x = -(z^2 + 2z^3 + \cdots + n!z^{n+1}) = z\ell \in z\hat{R}$.

1.5 Remark: Let R, M be as in 1.4. Then $\hat{R}$ is Noetherian by [8, Proposition 4.2]. Note that, as $x \in z\hat{R}, \hat{M} = z\hat{R} + \theta\hat{R}$. Let P, Q be the prime ideals zR and $xR + zR$, respectively, of R. Let $S = R/P, N = M/P$. As $z\hat{R}$ is closed in $\hat{R}$, see 1.3, $\hat{R}/z\hat{R}$ is isomorphic to the N-adic completion $\hat{S}$ of S. But $x + P \in \bigcap_{n\geq 1} N^n$ by [4, Proposition 1(iii)] so $\hat{S} \simeq k[[\theta]]$. Thus $z\hat{R}$ is a prime ideal and $\hat{R}$ is, in the sense of [**13**], a two dimensional regular local ring (it is not difficult to check that z is regular in $\hat{R}$). Observe that $P\hat{R} = Q\hat{R}$ and that $\hat{M}$ has height two whereas M has height three. Similar behaviour has been noted in [2]. In fact $\hat{R}$ can be identitied with the completion of its subring $T = k[z][\theta; z^2\partial/\partial z]$ at the maximal ideal $zT + \theta T$, with $x = -(z^2 + 2z^3 + \cdots)$. This consists of formal power series in z and θ subject to the homogeneous relation $\theta z = z\theta + z^2$.

1.6 QUESTION: *Let R be Noetherian with a maximal ideal M generated by a normalizing sequence $a_1, a_2, \ldots, a_s$. In view of 1.5, it would be interesting to know whether, in $\hat{R}, \hat{M}$ is generated by a normalizing sequence consisting of those a_i for which $a_iM = Ma_i + (a_1R + \cdots + a_{i-1}R)$. A positive answer to Question 1.3 would have some bearing on this question.*

2. Specification of the main example

Throughout this section, A will denote the commutative domain

$$k[X,Y,Z,T]/(XY - ZT) = k[x,y,z,t : xy = zt]$$

where k is a field of characteristic zero, X, Y, Z and T are indeterminates and x, y, z and t are their respective images in the factor ring. The element $XY - ZT$ is annihilated by the k-derivation $Z^2\partial/\partial Z + (X + Z^2)\partial/\partial X + (-Y + Z^2)\partial/\partial Y + Z(X + Y - T)\partial/\partial T$ of $k[X, Y, Z, T]$ and so there is an induced k-derivation δ of A satisfying

$$\delta(z) = z^2, \quad \delta(x) = x + z^2, \quad \delta(y) = -y + z^2 \quad \text{and} \quad \delta(t) = z(x + y - t).$$

Let B be the localization of A at the set $\{z^i\}_{i \geq 1}$. Then $t = xyz^{-1}$ and $B = k[z, z^{-1}, x, y]$. The derivation δ extends to B by the quotient rule and we denote the extension by δ.

Let R be the ring $A[\theta; \delta]$, a Noetherian domain. Then R has a maximal ideal M generated by the normalizing sequence z, x, y, t, θ. We shall show that $\bigcap_{n \geq 1} M^n = 0$ and that z is a zero-divisor in the M-adic completion $\hat{R}$.

2.1. The first step in showing that $\bigcap_{n \geq 1} M^n = 0$ is to show that B is δ-simple and hence that every δ-stable ideal of A contains a power of z. The following result, due to Shamsuddin [11], is useful in this respect.

PROPOSITION: *Let S be a commutative domain, containing $\mathbb{Q}$, with a derivation δ such that S is δ-simple. Let $s, t \in S$. Extend δ to the polynomial ring $\delta[x]$ by setting $\delta(x) = sx + t$. Then $S[x]$ is δ-simple if $\delta(r) \neq sr + t$ for all $r \in S$.*

Proof: To our knowledge, the only sources for this result are the Leeds PhD theses of Shamsuddin [11] and Archer [1] so we give an outline of the proof. Suppose that there exists a non-zero δ-stable proper ideal J of $S[x]$ and let n

be the minimal degree in x of non-zero elements of J. The leading coefficients of elements of J of degree n, together with 0, form a δ-stable ideal of S so J must contain an element g of the form $x^n + ax^{n-1} + \cdots + b$. By minimality of $n, \delta(g) - nsg = 0$ and, comparing coefficients of $x^{n-1}, \delta(a) = sa - nt$, whence $\delta(r) = sr + t$ where $r = -a/n$. ∎

2.2 PROPOSITION: *The ring B is δ-simple and consequently every non-zero δ-stable ideal of A contains a power of z.*

Proof: Let $B_1 = k[z, z^{-1}]$ and $B_2 = B_1[x]$ so that $B = B_2[y]$. Observe that δ restricts to k-derivations, also denoted δ, of B_1 and B_2. It is easy to check that B_1 is δ-simple and that $r + z^2 \neq \delta(r)$ for all $r \in B_1$. By 2.1, B_2 is δ-simple.

Now suppose that B_1 has an element r satisfying $\delta(r) = -r + z^2$. Set $B_3 = k[x, z]$ and write $r = pz^{-m}$ where $p \in B_3, m \geq 0$ and if $m > 0, p \notin zB_3$. Then $\delta(r) = \delta(p)z^{-m} - mpz^{1-m}$ whence $\delta(p) - mpz = -p + z^{2+m}$, that is, $z^2\partial p/\partial z + (x + z^2)\partial p/\partial x - mpz = -p + z^{2+m}$. Passing to the factor ring $\overline{B}_3 = B_3/zB_3, \overline{x}\partial\overline{p}/\partial\overline{x} = -\overline{p}$ and it follows that $\overline{p} = 0$. Thus $p \in zB_3$ and so $m = 0$ and $r \in B_3$. Now let B_4 be the overring $k[x][[z]]$ of B_3 to which δ extends in an obvious way. Consider the element $q = z^2 - 2z^3 + 6z^4 - 24z^5 + \ldots$ of B_4. Then $\delta(q) = -q + z^2$ so $\delta(r - q) = q - r$. But it is easy to check, comparing coefficients in $k[x]$ of powers of z, that the only element b of B_4 satisfying $\delta(b) = -b$ is 0. Hence $r = q$. But $q \notin B_3$ so we have a contradiction and, by 2.1, B is δ-simple. The consequence is immediate. ∎

2.3. The following result, which has a routine proof, will allow us to check that $\bigcap_{n\geq 1} M^n = 0$ by passing to an overring.

PROPOSITION: *Let A, C be rings, let $f : A \to C$ be a ring homomorphism and let δ, γ be derivations of A, C respectively such that $\gamma f = f\delta$. Then*

(i) *the kernel of f is δ-stable;*

(ii) *if f is injective then it can be extended to an injective ring homomorphism $f : A[\theta; \delta] \to C[\theta; \gamma]$ with $f(\theta) = \theta$.*

2.4. We aim to apply 2.3 to the ring A and derivation δ specified in 2.0. Let C be the formal power series ring $k[[z]]$ and let γ be the k-derivation $z^2\partial/\partial z$ of C.

LEMMA: *There is an injective ring homomorphism $f : A \to C$ given by*

$$\begin{aligned} f(z) &= z; \\ f(x) &= -z^2 - 2z^3 - 6z^4 - 24z^5 - \cdots; \\ f(y) &= z^2 - 2z^3 + 6z^4 - 24z^5 + \cdots; \\ f(t) &= f(x)f(y)/z, \end{aligned}$$

and satisfying $\gamma f = f\delta$.

Proof: Clearly there is a ring homomorphism f as specified. Let $\beta = \gamma f - f\delta$. If C is viewed as an A-module by means of f in the usual way, then β is a k-derivation from A to C, as defined in [7, p190]. One easily checks that $\beta(x) = \beta(y) = \beta(z) = \beta(t) = 0$ and so, as these elements generate A as a k-algebra, $\beta = 0$. Thus $\gamma f = f\delta$. By 2.3(i), the kernel of f is δ-stable. Since $f(z^m) \neq 0$ for all m, it follows from 2.2 that f is injective. ∎

2.5 PROPOSITION: $\bigcap_{n \geq 1} M^n = 0$.

Proof: By 2.3(ii) and 2.4 the injective homomorphism f specified in 2.4 extends to an injective homomorphism $f : R \to S$ where $S = C[\theta; \gamma]$ and C, γ are as in 2.4. Let $N = zC + \theta C$, a maximal ideal of C. Since $\theta z - z\theta = z^2$ any element of N^n must be a linear combination of monomials $z^i\theta^j$ with $i + j \geq n$. Consequently $\bigcap_{n \geq 1} N^n = 0$. But $f(M) \subseteq N$ so, since f is injective, $\bigcap_{n \geq 1} M^n = 0$. ∎

2.6. PROPOSITION: *The regular normal element z of R becomes a zero-divisor in the M-adic completion $\hat{R}$.*

Proof: As in 1.4, $x = z\ell$ where $\ell = -(z + 2z^2 + \ldots + n!z^n + \ldots) \in \hat{R}$. But $xy = zt$ so $z(t + zy + 2zy^2 + 6z^3y + \ldots) = 0$. Routine calculations show that $M^2 \cap A$ is generated by x, y, z^2 and t^2. Thus $t \notin M^2$ and, since $z^iy \in M^2$ for $i \geq 1$, $t + zy + 2z^2y + 6z^3y + \cdots \neq 0$ in $\hat{R}$. ∎

2.7 Remark: As in 1.5, one can check that $\hat{R}$ is Noetherian, that $z\hat{R}$ is a prime ideal with $\hat{R}/z\hat{R} \simeq k[[t, \theta]]$ and that $\hat{M} = z\hat{R} + t\hat{R} + \theta\hat{R}$. With f, C, S and N as in 2.4 and 2.5, $f(M) \subseteq N$ so that there is an induced map $\hat{f} : \hat{R} \to \hat{S}$. The ring $\hat{S}$ is a domain by [8, proof of 4.3] and [6, Theorem 1] so $\hat{f}$ is not injective.

2.8 QUESTION: *The normalizing sequence generating M in our example is not regular, x not being regular modulo zR. It would be interesting to know whether the behaviour described in 2.5 and 2.6 can occur when M is generated by a regular normalizing sequence. It is not hard to construct such sequences in such a way that the first term becomes a zero-divisor in the completion but in all such examples which we have computed $\bigcap_{n\geq 1} M^n \neq 0$.*

2.9 Remark: It is interesting to contrast the situation here, where the first term in the normalizing sequence loses regularity, with that of [6]. There, a crucial point in the proof that if I is a completely prime ideal of R generated by a regular normalizing sequence and satisfying the Artin-Rees property then the I-adic completion of R is a domain is that the first term in the sequence of generators is regular in the completion.

2.10 Remark: Note that, in our example, $\delta(t) \in zA + xA + yA$. It follows that $P = zR + xR + yR + \theta R$ is a prime ideal of R with factor isomorphic to $k[t]$. Since $\bigcap_{n\geq 1} M^n = 0, \bigcap_{n\geq 1} P^n = 0$. The element z is also a zero-divisor in the P-adic completion of R. It can easily be checked that, although $\bigcap_{n\geq 1} M^n = 0, z$ is in the intersection of the symbolic powers of P.

2.11 Remark: Return to the example of 1.4. The method used in 2.5 can be applied to show that $\bigcap_{n\geq 1} M^n = 0$ in this ring also. Here the proof is much shorter with the δ- simplicity of B_2, established in the first few lines of the proof of 2.2, replacing that of B and with 2.3 applied with C, γ and f as in 2.4 but ignoring y and t. This approach is also applicable to the ring of [5, Section 3], where it gives a shorter proof of [5, Propostion 9].

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